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On mirror symmetry in three dimensional abelian gauge theories

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Abstract: We present an identity relating the partition function of $\mathcal{N} = 4$ supersymmetric QED to that of its dual under mirror symmetry. The identity is a generalized Fourier transform. Many known properties of abelian theories can be derived from this formula, including the mirror transforms for more general gauge and matter content. We show that $\mathcal{N} = 3$ Chern-Simons QED and $\mathcal{N} = 4$ QED with BF-type couplings are conformal field theories with exactly marginal couplings. Mirror symmetry acts on these theories as strong-weak coupling duality. After identifying the mirror of the gauge coupling (sometimes called the "magnetic coupling") we construct a theory which is *exactly* mirror — at all scales — to $\mathcal{N} = 4$ SQED. We also study vortex-creation operators in the large N_f limit.

Keywords: Field Theories in Lower Dimensions, Duality in Gauge Field Theories, Chern-Simons Theories, Supersymmetry and Duality.

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Contents

1. Introduction

Major advances in supersymmetric field theory and string theory in various dimensions have led to the understanding that it is common for apparently different quantum field theories to be quantum-mechanically equivalent. Two theories which are "dual" in this way may be thought of as two choices of variables in a path integral representation for the same generating functional. Not that such "duality relations" are new; the relation between position space and momentum space representations of quantum mechanical systems are of this type; the order-disorder-fermion representations of the Ising model, the identity of the sine-Gordon model and the Thirring model, and target-space duality in two-dimensional sigma-models are well known from two dimensions; and it has long been conjectured that $\mathcal{N}=4$ supersymmetric Yang-Mills theory in four dimensions is a conformal field theory with a duality symmetry. The developments in the last few years have provided vast amounts of circumstantial evidence for the latter conjecture and have shown that many different duality transformations exist in higher dimensions with as few as four supercharges (which is $\mathcal{N}=1$ supersymmetry in four dimensions and $\mathcal{N}=2$ supersymmetry in three.)

However, outside of a small number of examples — free field theories, some lattice models, and a few two-dimensional continuum field theories — we do not know the precise change of variables which would allow the transformation from one representation of a theory to a dual representation. In this paper we take a small step toward making the "mirror symmetry"[1] of three dimensional $\mathcal{N}=4$ supersymmetric abelian gauge theories explicit. First, focusing on the infrared behavior of these theories, where mirror symmetry is exact, we present a formula which captures the essence of the mirror symmetry transformation in the form of a generalized Fourier transform. This formula encodes most known results in abelian mirror symmetry in simple ways. Second, we use the formula to derive some new results. We consider $\mathcal{N}=3$ Chern-Simons theories [2, 3] and $\mathcal{N}=4$ theories with BF-type couplings [4] interacting with matter, and argue they flow in the infrared to lines of fixed points parameterized by the coefficient of the CS or BF term. As we will show, mirror symmetry maps these models to models of the same type while inverting the CS or BF coupling; the inversion of the CS coupling agrees with [5]. Third, after identifying the field theory origin of the mirror of the gauge coupling (the so-called "magnetic coupling"), we use our formula to suggest a mirror for $\mathcal{N}=4$ SQED which is valid at all energy scales, not just in the infrared. Finally, we discuss the construction of the vortex-creation operators in $\mathcal{N}=4$ SQED, and compute their dimension at large N_f.

2. Preliminaries

We work in Minkowski space with signature $(-++)$. The $\mathcal{N}=4$ superalgebra has eight supercharges which are doublets under $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SU}(2)_R\times\mathrm{SU}(2)_N$; the first factor is the Lorenz group, while the last two are R-symmetries. We use indices $\alpha,\beta; i,j; a,b$ for the indices of the defining representation of these three factors. The abelian gauge theories which are the subject of this paper describe the interaction of U(1) vector multiplets $\mathcal{V}$ and charged hypermultiplets $\mathcal{Q}$. In components the vector multiplet contains a gauge boson A_μ, a gaugino $\lambda^i_{a\alpha}$ and three real scalars $\Phi^{\{ij\}}$, while the hypermultiplet contains a doublet of complex scalars Q_a and a doublet of spinors $\psi_{i\alpha}$. Because $\mathcal{N}=4$ superspace is often inconvenient, we will use $\mathcal{N}=2$ superspace language. The hypermultiplet can be written as two $\mathcal{N}=2$ chiral superfields $Q,\tilde{Q}$ of charge $1,-1$. The $\mathcal{N}=4$ vector multiplet consists of an $\mathcal{N}=2$ real vector multiplet V whose lowest component is a real scalar and a chiral multiplet Φ whose lowest component is a complex scalar. In the $\mathcal{N}=2$ notation only the $\mathrm{U}(1)_N\subset\mathrm{SU}(2)_N$ R-symmetry is explicit; the superfield Φ has $\mathrm{U}(1)_N$ charge 2, while the rest of the $\mathcal{N}=2$ superfields are uncharged.

The $\mathcal{N}=4$ supersymmetry algebra has an idempotent outer automorphism which interchanges $\mathrm{SU}(2)_R$ and $\mathrm{SU}(2)_N$. This automorphism takes an ordinary vector multiplet, whose scalars transform as a triplet of $\mathrm{SU}(2)_N$, into a twisted vector multiplet [4], whose scalars transform as a triplet of $\mathrm{SU}(2)_R$. Similarly, one can define a twisted hypermultiplet whose bosonic fields form an $\mathrm{SU}(2)_N$ doublet. Field and superfield constituents of twisted $\mathcal{N}=4$ multiplets will be distinguished with a hat, e.g. $\hat{\Phi}$ for the chiral part of the twisted vector multiplet. Note that $\hat{\Phi}$ has $\mathrm{U}(1)_N$ charge 0, while $\hat{Q}$ and $\hat{\tilde{Q}}$, the chiral constituents of the twisted hypermultiplet, have $\mathrm{U}(1)_N$ charge 1.

In three dimensions a photon is the electric-magnetic dual of a scalar. The scalar is periodic if the gauge group is compact, and shifting it by a constant is a symmetry of the classical theory. This duality transformation takes a free $\mathcal{N}=4$ vector multiplet into a twisted hypermultiplet with target space $\mathbb{R}^3 \times \mathbf{S}^1$ or $\mathbb{R}^4$, depending on whether the gauge group is compact or not. Similarly, a free twisted vector multiplet is the electric-magnetic dual of an ordinary hypermultiplet.

Yet another type of $\mathcal{N}=4$ multiplet is a linear multiplet (which also has a twisted version). Linear multiplets are important because they contain conserved currents. An $\mathcal{N}=4$ linear multiplet consists of an $\mathcal{N}=2$ linear multiplet Σ (a real superfield satisfying $D^2\Sigma=\bar{D}^2\Sigma=0$) and an $\mathcal{N}=2$ chiral superfield Π with $\mathrm{U}(1)_N$ charge 2. The field strength of a vector multiplet F resides in a linear multiplet with $\Sigma = iD\bar{D}V$ and $\Pi=\Phi$. The conserved current in this case is ${}^*F$ where $*$ is the Hodge star; the conservation of ${}^*F$ is a consequence of the Bianchi identity $dF=0$. The charge associated to this current is the generator of the shift symmetry of the dual photon mentioned above. The Noether current associated with the flavor symmetries of a twisted hypermultiplet also resides in an $\mathcal{N}=4$ linear multiplet; in this case $\Sigma = \hat{Q}^\dagger\hat{Q} - \hat{\tilde{Q}}\hat{\tilde{Q}}^\dagger$, $\Pi = \hat{\tilde{Q}}\hat{Q}$. Conversely, the topological current ${}^*\hat{F}$ associated with a twisted vector multiplet and the flavor current of an ordinary hypermultiplet reside in a twisted linear multiplet. The latter consists of an $\mathcal{N}=2$ linear multiplet $\hat{\Sigma}$ and a chiral multiplet $\hat{\Pi}$ with $\mathrm{U}(1)_N$ charge 0.

The action of an $\mathcal{N}=4$ theory of hypermultiplets and abelian vector multiplets contains kinetic terms for the hypermultiplets

$$S_H(\mathcal{Q},\mathcal{V}) = -\int d^3x\ d^4\theta\ \left(Q^\dagger e^{2V}Q + \tilde{Q}^\dagger e^{-2V}\tilde{Q}\right) - \left[\int d^3x\ d^2\theta\ i\sqrt{2}\Phi Q\tilde{Q} + \text{c.c.}\right], \tag{2.1}$$

which include scalar kinetic terms $-|D_\mu Q|^2 - |D_\mu \tilde{Q}|^2$, and kinetic terms for the vector multiplets

$$\frac{1}{g^2}S_V(\mathcal{V}) = \frac{1}{g^2}\int d^3x\ d^4\theta\ \left\{\frac{1}{4}\Sigma^2 - \Phi^\dagger\Phi\right\}, \tag{2.2}$$

which include $\frac{1}{4g^2}F_{\mu\nu}^2$. One can also add mass terms for the hypermultiplets

$$S_m(\mathcal{Q}) = -\int d^3x\ d^4\theta \left(Q^\dagger e^{-2im_r\theta\bar\theta}Q + \tilde{Q}^\dagger e^{2im_r\theta\bar\theta}\tilde{Q}\right) - \left[\int d^3x\ d^2\theta\ mQ\tilde{Q} + \text{c.c.}\right] \tag{2.3}$$

and Fayet-Iliopoulos (FI) terms for the vector multiplets

$$S_{FI}(\mathcal{V}) = \frac{\xi_r}{\pi}\int d^3x\ d^4\theta\ V - \left[\frac{i\xi}{2\pi}\int d^3x\ d^2\theta\ \Phi + \text{c.c.}\right]. \tag{2.4}$$

Here $m_r \in \mathbb{R}$ and $m \in \mathbf{C}$ together form an $\mathrm{SU}(2)_N$ triplet while $\xi_r \in \mathbb{R}$ and $\xi \in \mathbf{C}$ form an $\mathrm{SU}(2)_R$ triplet. These building blocks suffice to construct the most general renormalizable action containing only ordinary hypermultiplets and vector multiplets. The most general renormalizable action for twisted fields is obtained by putting hats over all fields in (2.1)–(2.4).

If the mass terms and the FI terms are zero, the $\mathcal{N} = 4$ action has $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_N$ R-symmetry, as well as two discrete symmetries which we call P and CP. To define these discrete symmetries we need to recall how parity transformation acts on Majorana spinors in three dimensions. Let us define parity as a reflection of one of the spatial coordinates, say $x^1 \to x'^1 = -x^1$. To ensure parity-invariance of the Dirac equation we must transform spinors according to $\psi \to R\psi$, where the two-by-two matrix R satisfies $R^TR = 1,\ R^T\gamma^0\gamma^1R = -\gamma^0\gamma^1,\ R^T\gamma^0\gamma^2R = \gamma^0\gamma^2$. Then parity acts on $\mathcal{N} = 2$ superspace via $x'^0 = x^0, x'^1 = -x^1, x'^2 = x^2, \theta' = R\theta$. The chiral superspace measure $d^2\theta$ is parity-odd, while $d^4\theta$ is parity-even. We define P as a transformation which acts on $\mathcal{N} = 2$ superfields via

$$\begin{aligned} V'(x',\theta') &= V(x,\theta), \quad \Phi'(x',\theta') = -\Phi(x,\theta)\,, \\ Q'(x',\theta') &= Q(x,\theta), \quad \tilde{Q}'(x',\theta') = \tilde{Q}(x,\theta)\,. \end{aligned} \tag{2.5}$$

The transformation CP is defined by

$$\begin{aligned} V'(x',\theta') &= -V(x,\theta), \quad \Phi'(x',\theta') = \Phi(x,\theta)\,, \\ Q'(x',\theta') &= \tilde{Q}(x,\theta), \quad \tilde{Q}'(x',\theta') = -Q(x,\theta)\,. \end{aligned} \tag{2.6}$$

We call these transformations P and CP because the gauge field behaves as a polar vector with respect to P and as an axial vector with respect to CP. It easy to check that when masses and FI terms are absent, the action is both P and CP-invariant. Mass terms break P, while FI terms break CP. To define P and CP for twisted multiplets we simply put hats over all fields in eqs. (2.5) and (2.6).

Let us recall how mirror symmetry works for $\mathcal{N} = 4$ abelian gauge theories without twisted fields [1]. If we set the hypermultiplet masses and FI couplings to zero, then the only mass scale in these theories is g^2, and they are believed to flow to nontrivial superconformal fixed points in the infrared, where the scale g^2 is washed out. Each of these superconformal fixed points has a dual description using the duality mapping known as mirror symmetry [1]. Under mirror symmetry, electrically charged particles and Abrikosov vortex solitons are exchanged. The mirror theory is a twisted abelian gauge theory, i.e. the fundamental degrees of freedom live in twisted hypermultiplets and twisted vector multiplets. The Higgs branch of one theory is the Coulomb branch of its mirror; similarly, the mass terms for the hypermultiplets are mirror to the FI terms for the twisted vector multiplets (which determine the masses of vortices.) The mapping of flavor symmetries is generally complicated. The U(1) currents from abelian subgroups of hypermultiplet flavor symmetries are mapped to the U(1) currents ${}^*\hat{F}$ [1]. The off-diagonal currents of the flavor symmetries are not seen semiclassically and will not be discussed below.

For example, the mirror of $\mathcal{N}=4$ U(1) with N_f flavors [we will refer to this theory as SQED-N_f] is a twisted $\mathrm{U}(1)^{N_f-1}$ gauge theory with N_f twisted hypermultiplets $\hat{Q}_p, \hat{\tilde{Q}}_p, p=1,\ldots,N_f$, where $\hat{Q}_p$ has charge $+1$ under the p^{th} U(1) factor and charge -1 under the $(p-1)^{th}$ U(1) factor [1]. The topological current ${}^*F$ of $\mathcal{N}=4$ SQED is mirror to the Noether current which generates a U(1) global symmetry transformation $\hat{Q}_p \to e^{i\alpha}\hat{Q}_p, \hat{\tilde{Q}}_p \to e^{-i\alpha}\hat{\tilde{Q}}_p, p=1,\ldots,N_f$. It is convenient to choose the normalization in which $\hat{Q}$ has charge $1/N_f$ under this global U(1); then all gauge invariant operators in the $\mathrm{U}(1)^{N_f-1}$ theory have integer global U(1) charges. Note that the case $N_f=2$ is special, since the mirror theory is isomorphic to the original one [1].

It is also interesting to consider theories which contain both ordinary and twisted $\mathcal{N}=4$ multiplets. A natural way to couple twisted and ordinary vector multiplets is by means of an $\mathcal{N}=4$ BF term [4]. It appears that this is the only way to couple twisted and ordinary fields without introducing operators of dimension higher than three. The $\mathcal{N}=4$ BF term has the following form:

$$S_{BF}(\hat{\mathcal{V}},\mathcal{V}) = \frac{1}{2\pi}\int d^3x\; d^4\theta\; V\hat{\Sigma} - \left[\frac{1}{2\pi}\int d^3x\; d^2\theta\; i\Phi\hat{\Phi} + \text{c.c.}\right]. \tag{2.7}$$

Its component form in the Wess-Zumino gauge is given by

$$S_{BF} = \frac{1}{2\pi}\left(-\frac{1}{2}\epsilon^{mnp}A_m\hat{F}_{np} + \Phi^{\{ij\}}\hat{D}_{\{ij\}} + \hat{\Phi}^{\{ab\}}D_{\{ab\}} + i\lambda^i_{a\alpha}\hat{\lambda}^{a\alpha}_i\right). \tag{2.8}$$

Here $D_{\{ab\}}$ and $\hat{D}_{\{ij\}}$ are the auxiliary fields of the ordinary and twisted vector multiplets respectively. The authors of [4], who were the first to construct the $\mathcal{N}=4$ BF coupling, observed that both ordinary and twisted multiplets were required. Noting the analogy with two dimensions, they correctly conjectured the existence of a mirror symmetry which would exchange these multiplets. We will see in section 3 that the BF interaction lies at the heart of the mirror transform.

The BF term gives a gauge-invariant mass to both $\mathcal{V}$ and $\hat{\mathcal{V}}$. It also breaks P and CP. Certain discrete symmetries remain unbroken, however. Namely, a transformation which acts as P (CP) on the ordinary fields and as CP (P) on the twisted fields is still a symmetry.

When a BF term is present in the action, one can dualize either a twisted vector multiplet, or an ordinary one, but not both of them simultaneously. Also, in the presence of the BF term the shift symmetry of the dual photon is gauged. This will be discussed in more detail in section 4.

We will also need gauge-fixing terms. Their explicit form is unimportant for our purposes. For example, one can use an $\mathcal{N}=2$–covariant version of Landau gauge:

$$S_{GF}(\mathcal{V}) = \int d^3x \left(\int d^2\theta \; \Psi \bar{D}^2 V + \text{c.c.} \right), \tag{2.9}$$

where Ψ is a chiral superfield serving as a Lagrange multiplier. This particular gauge-fixing term breaks $\mathcal{N}=4$ supersymmetry down to $\mathcal{N}=2$ but the correlators of gauge-invariant quantities remain $\mathcal{N}=4$-supersymmetric.

As in [1], one can prove nonrenormalization theorems for various branches of the moduli space. In particular, the metric on the Higgs branch, where ordinary hypermultiplets have VEVs, does not depend on the gauge coupling of the ordinary vector multiplets. Similarly, the metric on the twisted Higgs branch, where twisted hypermultiplets have VEVs, is unaffected by the twisted gauge coupling. On the other hand, in the presence of the BF term the metric on the Higgs branch does depend on the twisted gauge coupling. We will see this explicitly in section 4.

3. The mirror transform is a fourier transform

It has been known for some time that most known results of abelian mirror symmetry can be derived from the properties of $\mathcal{N}=4$ supersymmetric U(1) gauge theory with a single charged hypermultiplet (an electron, a positron and their scalar partners.) This theory, which we will call SQED-1, flows from weak coupling in the ultraviolet to strong coupling in the infrared, where it becomes a conformal field theory (CFT) which we will refer to as CFT-1. The fundamental result of mirror symmetry is that CFT-1 is equivalent to a Gaussian theory [6] — namely, a free twisted hypermultiplet.

SQED-1 has a single abelian global symmetry whose current is ${}^*F$. The associated charge, the integrated magnetic flux, is the vortex number. We may couple this current and its superpartners to a background twisted vector multiplet $\hat{\mathcal{V}}$ through a BF-type interaction (2.7). Then the generating functional for correlation functions of the current multiplet in SQED-1 can be written as

$$Z_{\mathrm{SQED-1}}[\hat{\mathcal{V}}] = \int \mathcal{DV}\,\mathcal{DQ}\;\exp\left(\frac{i}{g^2}S_V(\mathcal{V}) + iS_{GF}(\mathcal{V}) + iS_{BF}(\hat{\mathcal{V}},\mathcal{V}) + iS_H(\mathcal{Q},\mathcal{V})\right). \tag{3.1}$$

We define this functional integral as the sum of its expansion in powers of g^2. Since the theory is abelian, there are no instanton corrections. Power counting and symmetries imply that there are no divergences in this expansion, so no counterterms are needed in (3.1). Since g is the only available scale, the perturbative expansion is actually an expansion in powers of g^2/p where p is momentum. To obtain the infrared CFT, one needs to resum the perturbative series and take the limit $g \to \infty$. Applying this limit formally to (3.1) we obtain the expression

$$Z_{\mathrm{CFT-1}}[\hat{\mathcal{V}}] = \int \mathcal{DV}\,\mathcal{DQ}\; e^{iS_{GF}(\mathcal{V})+iS_{BF}(\hat{\mathcal{V}},\mathcal{V})+iS_H(\mathcal{Q},\mathcal{V})}\;. \tag{3.2}$$

As mentioned above, CFT-1 is equivalent to a theory of a free twisted hypermultiplet $\hat{\mathcal{Q}}$, with the field strength of $\mathcal{V}$ being mapped to the abelian U(1) flavor current of the twisted hypermultiplet. The appropriate path integral is

$$Z_{\hat{\mathcal{Q}}}[\hat{\mathcal{V}}] = \int \mathcal{D}\hat{\mathcal{Q}}\; e^{iS_H(\hat{\mathcal{Q}},\hat{\mathcal{V}})}\;. \tag{3.3}$$

The statement of mirror symmetry is $Z_{\hat{\mathcal{Q}}}[\hat{\mathcal{V}}] = Z_{\mathrm{CFT-1}}[\hat{\mathcal{V}}]$. The integrals over the hypermultiplets are quadratic and give a superdeterminant of the supersymmetric Laplacian $\mathcal{K}$ on flat $d = 3$ $\mathcal{N} = 4$ superspace. Using this, we may write the equivalence of these two generating functionals in the following suggestive form:

$$\mathrm{Sdet}\left(\mathcal{K}[\hat{\mathcal{V}}]\right) = \int \mathcal{DV}\; e^{iS_{GF}(\mathcal{V})} e^{iS_{BF}(\hat{\mathcal{V}},\mathcal{V})}\;\mathrm{Sdet}\left(\mathcal{K}[\mathcal{V}]\right)\;. \tag{3.4}$$

Mirror symmetry between $\mathcal{N} = 4$ SQED-1 and the theory of a free twisted hypermultiplet is thus related to the invariance of the superdeterminant under a Fourier transform with respect to the background fields. Note that this is highly non-trivial, as neither superdeterminant is Gaussian.

The relation (3.4) encapsulates many known properties of mirror symmetry and allows them to be rederived using elementary manipulations. We list a few examples here.

3.1 Inverse and repeated Fourier transforms

To invert the functional Fourier transform (FFT) we multiply both sides of (3.4) by $\exp[-iS_{BF}(\mathcal{V}',\hat{\mathcal{V}}) - iS_{GF}(\hat{\mathcal{V}})]$, where $\mathcal{V}'$ is another background vector multiplet, and integrate both sides of (3.4) over $\hat{\mathcal{V}}$. The physical meaning of these manipulations is that gauging a global symmetry in one theory corresponds to removing a gauge symmetry (sometimes called "ungauging") using a BF coupling in its mirror. If instead we apply the FFT to $\mathrm{Sdet}(K[\mathcal{V}])$ twice, we get $\mathrm{Sdet}(K[-\mathcal{V}])$, implying that the fourth power of the FFT is the identity transformation. (This is also true for the ordinary Fourier transform on the space of C^∞ functions of rapid decrease.) In the string-theoretic approach of [7] to mirror symmetry, the mirror transform is effected by a generator S of $\mathrm{SL}(2,Z)$ which also satisfies $S^4 = 1$.

3.2 Mapping of operators

As discussed in sec. 2, mirror symmetry maps hypermultiplet masses to Fayet-Iliopoulos terms, hypermultiplet flavor currents to topological currents, and the Higgs branch of moduli space to the Coulomb branch. These mappings can be easily seen in eq. (3.4). A free hypermultiplet has a Higgs branch parameterized by $\langle \hat{Q}\tilde{\hat{Q}} \rangle$ and $\langle \hat{Q}^\dagger \hat{Q} - \tilde{\hat{Q}}^\dagger \tilde{\hat{Q}} \rangle$, while CFT-1 has a Coulomb branch with coordinates $\langle \Phi \rangle$, $\langle \Sigma \rangle$. That these branches are exchanged is made clear by taking derivatives of the two sides of (3.4) with respect to $\hat{\Phi}$, $\hat{V}$. Similarly, a constant expectation value for $\hat{\Phi}$ gives a mass to the hypermultiplet $\hat{\mathcal{Q}}$ while inducing a Fayet-Iliopoulos coupling for $\mathcal{V}$. The background gauge field $\hat{A}_\mu$ couples to the topological current of CFT1 and to the flavor current of the free hypermultiplet.

3.3 The convolution theorem

The inverse FFT and the convolution theorem may be applied to derive mirror symmetry in all other abelian $\mathcal{N} = 4$ theories. For example, to study (twisted) SQED-N_f, with (twisted) hypermultiplets $\hat{\mathcal{Q}}_i$ of charge 1, one raises both sides of eq. (3.4) to the power N_f and then integrates over $\hat{\mathcal{V}}$. On the left-hand side one gets the partition function of the twisted SQED-N_f. On the right-hand side, the integration over $\hat{\mathcal{V}}$ removes the vector multiplet which couples equally to all N_f hypermultiplets. The remaining $N_f - 1$ vectors and N_f hypermultiplets form the $\mathrm{U}(1)^{N_f-1}$ theory described in sec. 2. Similar manipulations allow one to find the mirror of an arbitrary abelian $\mathcal{N} = 4$ theory. The results agree with [8].

3.4 $\mathcal{N} = 2$ mirror symmetry

$\mathcal{N} = 2$ SQED with two oppositely charged chiral superfields $Q, \tilde{Q}$ can be obtained from $\mathcal{N} = 4$ SQED-1 by coupling the latter to a neutral chiral superfield S via the interaction $\int d^2\theta\, S\Phi$, which makes both S and Φ massive [9, 10]. We may identify the chiral field $\hat{\Phi}$ int he twisted vector multiplet $\hat{\mathcal{V}}$ with S and integrate over $\hat{\Phi}$ on both sides of (3.4) with weight $\exp(-i/h \int d^4\theta\, \hat{\Phi}^\dagger \hat{\Phi})$. In our normalization $\hat{\Phi}$ has engineering dimension 1, therefore h is a parameter of dimension 1. The right-hand side becomes the partition function of a theory whose infrared (large h) limit is the same as the infrared limit of $\mathcal{N} = 2$ SQED-1. The left-hand side is a partition function of an $\mathcal{N} = 2$ theory of three chiral superfields $\hat{Q}, \tilde{\hat{Q}}, \hat{\Phi}$ coupled via the superpotential $W = \hat{\Phi}\hat{Q}\tilde{\hat{Q}}$. These two theories were shown to be mirror in [9, 10]. Mirror symmetry in all other $\mathcal{N} = 2$ abelian theories can again be derived using the convolution theorem and the inverse Fourier transform. In all these theories there are no ultraviolet divergences (if regularization preserves all the symmetries), so our formal manipulations are presumably justified.

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4. $\mathcal{N}=4$ theories with BF couplings

In this section we study $\mathcal{N}=4$ theories which contain both ordinary and twisted fields coupled via a BF term eq. (2.7). These theories apparently have not been considered in the literature. We will show that BF couplings are exactly marginal. They parameterize manifolds of conformal field theories, in analogy to Maxwell couplings in finite $d=4$ $\mathcal{N}=2$ supersymmetric gauge theories. As in the four-dimensional case, weakly coupled theories are found near certain boundary points of these manifolds, with the inverse BF couplings serving as expansion parameters for a finite perturbation series around a free theory. Mirror symmetry acts on these manifolds by exchanging strongly coupled SCFTs with weakly coupled ones.

To be concrete, let us consider a copy of $\mathcal{N}=4$ SQED-1 and a copy of its twisted version, coupled via a BF-term with coefficient k. The classical action of this theory is

$$\frac{1}{g^2}S_V(\mathcal{V})+\frac{1}{\hat{g}^2}S_V(\hat{\mathcal{V}})+S_H(\mathcal{Q},\mathcal{V})+S_H(\hat{\mathcal{Q}},\hat{\mathcal{V}})+kS_{BF}(\mathcal{V},\hat{\mathcal{V}})\,. \tag{4.1}$$

The vector multiplets become topologically massive and therefore both the twisted and ordinary Coulomb branches are lifted. The classical moduli space of this theory consists of a Higgs branch and a twisted Higgs branch parameterized by the ordinary and twisted hypermultiplet vacuum expectation values, respectively. These branches intersect at a single point (the origin.) To determine the metric we must solve the D-flatness conditions modulo gauge transformations. The D-flatness conditions for the Higgs branch are

$$\mathcal{Q}^\dagger\sigma^p\mathcal{Q}=\frac{k}{2\pi}\hat{\Phi}^p\,,\qquad p=1,2,3\,. \tag{4.2}$$

Note that the lowest component of the twisted vector multiplet plays the role of the dynamical Fayet-Iliopoulos term. We expect that eqs. (4.2) can be interpreted as moment map equations for a hyperkähler quotient [11] (see also [12] for a review).

To see that this is indeed the case, recall that we can dualize the twisted photon $\hat{A}$ into a scalar $\hat{\tau}$. $\hat{\tau}$ can be combined with $\hat{\Phi}^p$ into a quaternion

$$w=\frac{\hat{g}^2\hat{\tau}}{\pi\sqrt{2}}+i\sigma^p\hat{\Phi}^p\,,$$

where $\hat{g}$ is the twisted gauge coupling. In terms of w the kinetic energy of the twisted vector multiplet takes the form

$$\frac{1}{2\hat{g}^2}|\partial w|^2\,.$$

The metric $|dw|^2/(2\hat{g}^2)$ is, up to an overall factor, the standard hyperkähler metric on $\mathbb{R}^4$ (or $\mathbb{R}^3\times\mathbf{S}^1$ if we take the gauge group to be compact.) We can also think of the complex doublet $\mathcal{Q}$ as a quaternion, with metric $|d\mathcal{Q}|^2$. Under a constant gauge transformation from the untwisted U(1), $\mathcal{Q}$ transforms as $\mathcal{Q}\to\mathcal{Q}e^{i\sigma_1\alpha}$. Less trivially,

w transforms as $w \to w - k\hat{g}^2\alpha/(\pi\sqrt{2})$, as explained below. The hyperkähler moment equations for this transformation are precisely (4.2). It is well known that this hyperkähler quotient yields the Taub-NUT metric [12], and thus the Higgs branch is the Taub-NUT space. This space can be thought of as a circle fibered over $\mathbb{R}^3$. The Taub-NUT metric depends on a single parameter which sets the asymptotic radius of this circle. In the present case this radius is $k\hat{g}$. Identical arguments show that the twisted Higgs branch is also a Taub-NUT space, the asymptotic radius being kg. Note that these results are in agreement with the nonrenormalization theorem stated in section 2, which says that the metric on the Higgs branch (resp. twisted Higgs branch) does not depend on the gauge coupling (resp. twisted gauge coupling).

Let us recall why the dual photon $\hat{\tau}$ transforms additively, $\hat{\tau} \to \hat{\tau} - k\alpha$, under a constant gauge transformation. In short, the BF coupling $A \wedge \hat{F}$ can be interpreted as a coupling of the gauge field A to a topological current ${}^*\hat{F}$ generating the shift of the dual photon. This means that the shift symmetry is gauged, A being the corresponding gauge field, so a gauge transformation of A must be accompanied by a shift of $\hat{\tau}$. A more detailed argument goes as follows. In order to dualize the twisted gauge field $\hat{A}$ we need to treat its field strength $\hat{F}$ as an unconstrained 2-form and impose the Bianchi identity $d\hat{F} = 0$ via a Lagrange multiplier $\hat{\tau}$. Then the action takes the form

$$-\frac{1}{4\pi}\epsilon^{mnp}\hat{F}_{mn}(kA_p + \partial_p\hat{\tau}) + \cdots ,$$

where dots denote terms which are manifestly invariant with respect to gauge transformations $A \to A + d\alpha$. The action will be invariant if we also transform $\hat{\tau}$ as $\hat{\tau} \to \hat{\tau} - k\alpha$.

Our discussion of the metric was classical, but one can show that there are no quantum-mechanical corrections. Indeed, supersymmetry tells us that the metric is hyperkähler, and we also know that it has $\mathrm{SU}(2)_R$ isometry ($\mathrm{SU}(2)_N$ for the twisted Higgs branch) which rotates the three complex structures. This, together with the known asymptotic behavior, uniquely determines the metric [13].

In the infrared limit we must take both gauge couplings to infinity, and then the moduli space becomes a pair of $\mathbb{R}^4$'s intersecting at the origin. For infinite gauge couplings the one-particle poles in the propagators of the vector multiplets move to infinity; nevertheless the vector multiplets cannot be ignored, since the BF term remains and induces a nontrivial interaction between the ordinary and twisted hypermultiplets, whose strength depends on k. The theory at the origin of the moduli space is a nontrivial CFT (as the moduli space is not smooth there) with $\mathcal{N} = 4$ SUSY and unbroken $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_N$ symmetry.

For $k \to \infty$ the vector multiplets decouple, so the CFT at the origin becomes a direct sum of a free hypermultiplet and a free twisted hypermultiplet. It is straightforward to set up perturbation theory in $1/k$. Using the approach of Refs. [14, 15] it is easy to show that the coefficient of the BF-term k is not renormalized, so the CFT at the origin is an exactly marginal deformation of the theory with $k = \infty$, i.e. of a free theory. In fact, the high degree of supersymmetry ensures that there are no ultraviolet divergences in this expansion. The dimension of any operator can be computed as a power series in $1/k$. The dimensions of operators in short representations of the superconformal algebra are determined by their $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_N$ quantum numbers: $\Delta = j_R + j_N$ for scalar primary operators, where j_R and j_N are the $\mathrm{SU}(2)_R$ and $\mathrm{SU}(2)_N$ spins [16]. The dimensions of operators in long superconformal multiplets depend on k, in general.

In the opposite limit, $k \to 0$, ordinary and twisted multiplets do not couple to each other and the theory flows to a direct sum of CFT-1 and twisted CFT-1. Note that this theory is mirror to the theory at $k \to \infty$. One may conjecture that more generally the CFT at large k and the CFT at small k are mirror to each other. To show that this is indeed the case, consider the following generating functional for the CFT with BF coupling k:

$$Z_k[\mathcal{U}, \hat{\mathcal{U}}] = \int \mathcal{D}\mathcal{V}\, \mathcal{D}\hat{\mathcal{V}}\ \mathrm{Sdet}\left(\mathcal{K}[\hat{\mathcal{V}}]\right) \mathrm{Sdet}\left(\mathcal{K}[\mathcal{V}]\right) \exp\Big(ikS_{BF}(\hat{\mathcal{V}}, \mathcal{V}) + \\ + iS_{GF}(\mathcal{V}) + iS_{GF}(\hat{\mathcal{V}}) + iS_{BF}(\hat{\mathcal{V}}, \mathcal{U}) + iS_{BF}(\hat{\mathcal{U}}, \mathcal{V})\Big). \tag{4.3}$$

Differentiating Z_k with respect to $\mathcal{U}$ and $\hat{\mathcal{U}}$ generates all the correlations functions of ordinary and twisted vector multiplets. Now we substitute the right-hand side of (3.4) for $\mathrm{Sdet}\left(\mathcal{K}[\mathcal{V}]\right)$ and $\mathrm{Sdet}\left(\mathcal{K}[\hat{\mathcal{V}}]\right)$ and perform Gaussian integrals over $\mathcal{V}$ and $\hat{\mathcal{V}}$. The result turns out to be

$$Z_{-1/k}\left[-\frac{1}{k}\mathcal{U}, -\frac{1}{k}\hat{\mathcal{U}}\right] \exp\left(-\frac{i}{k} S_{BF}(\hat{\mathcal{U}}, \mathcal{U})\right). \tag{4.4}$$

This means that the connected correlation functions of $\mathcal{V}$ and $\hat{\mathcal{V}}$ in CFTs with BF couplings k and $-1/k$ are related by a trivial rescaling. The two-point functions in addition differ by a contact term.

Our discussion can be easily generalized to theories with $N_f > 1$ and larger gauge groups. For example, take N_f copies of CFT-1, each with a hypermultiplet of charge 1. Take a similar set of copies of twisted CFT-1, and couple the vector multiplets to the twisted vector multiplets via a BF term, giving the action

$$S_H(\mathcal{Q}_i, \mathcal{V}_i) + S_H(\hat{\mathcal{Q}}_i, \hat{\mathcal{V}}_i) + \sum_{i,j} k_{ij} S_{BF}(\mathcal{V}_i, \hat{\mathcal{V}}_j).$$

This yields a manifold of $\mathcal{N} = 4$ SCFTs parameterized by the matrix k. Mirror symmetry acts on this manifold by $k \to -k^{-1}$. When k is nondegenerate, the Coulomb branches are lifted. The metrics of the Higgs and twisted Higgs branches can be computed as in the $N_f = 1$ case above and, for finite Maxwell couplings, turn out to be of the Lindström-Roček/Lee-Weinberg-Yi (LR/LWY) type [17, 12].

In summary, $\mathcal{N} = 4$ theories with BF couplings are similar in many respects to certain finite $N = 2$ theories in four dimensions. Both types of theories are finite in perturbation theory and have exactly marginal couplings (real in $d = 3$ and complex

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in $d = 4$) which are acted upon by a duality transformation. In $d = 3$ this duality is mirror symmetry, while in $d = 4$ it is electric-magnetic duality.

5. $\mathcal{N} = 3$ theories with Chern-Simons couplings

It is believed impossible to write down an $\mathcal{N} = 4$ supersymmetric Chern-Simons (CS) action coupled to matter. However, an $\mathcal{N} = 3$ CS action exists [2, 3]. One simply identifies the ordinary and twisted vector multiplets appearing in the BF action (2.7). This identification obviously breaks $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_N$ symmetry down to its diagonal subgroup $\mathrm{SU}(2)_D$ and, less obviously, breaks $\mathcal{N} = 4$ SUSY down to $\mathcal{N} = 3$. Under $\mathrm{SU}(2)_D$ the four supercharges of $\mathcal{N} = 4$ decompose as $\mathbf{1} + \mathbf{3}$. The identification breaks the singlet while the triplet survives. We will see the properties of these theories are very similar to those considered in the previous section.

Let us briefly review $\mathcal{N} = 3$ SUSY theories. The basic multiplets are the hypermultiplet and the vector multiplet. The hypermultiplet contains an $\mathrm{SU}(2)_D$-doublet of scalars Q_a and an $\mathrm{SU}(2)_D$-doublet of spinors ψ_a. The vector multiplet contains a triplet of real scalars $\Phi^{\{ab\}}$, a triplet of spinors $\lambda^{\{ab\}}$, a singlet spinor λ^0, and a gauge boson A_μ. An $\mathcal{N} = 3$ action for hypermultiplets automatically has $\mathcal{N} = 4$ SUSY. In particular an $\mathcal{N} = 3$ sigma-model must have a hyperkähler target space. If restrict ourselves to renormalizable theories, then the most general $\mathcal{N} = 3$ action for hypermultiplets interacting with abelian vector multiplets is the sum of the $\mathcal{N} = 4$ action and the $\mathcal{N} = 3$ Chern-Simons term for the vector multiplets

$$\sum_{i,j} k_{ij} S_{CS}(\mathcal{V}_i, \mathcal{V}_j) = \sum_{i,j} \frac{k_{ij}}{4\pi} \left(\int d^3x d^4\theta \; \Sigma_i V_j - \left[\int d^3x d^2\theta \; \Phi_i \Phi_j + \text{c.c.} \right] \right). \quad (5.1)$$

Here k_{ij} is a real symmetric matrix.[1] In what follows the parameters k_{ij} will be referred to as Chern-Simons couplings, while the coefficients of the Maxwell terms will be called gauge couplings, as before. Note that the $\mathcal{N} = 4$ theories with BF couplings considered above form a subset of the set of $\mathcal{N} = 3$ CS theories.

$\mathcal{N} = 3$ gauge theories have in general both Coulomb and Higgs branches. When k_{ij} is nondegenerate, all vector multiplets become massive and the Coulomb branch is lifted, while the Higgs branch remains (if the number of hypermultiplets exceeds the number of vector multiplets.) Quantum corrections cannot lift this branch but, unlike the $\mathcal{N} = 4$ case, they can modify its metric. The form of the quantum corrections to the metric is tightly constrained by the requirement that the metric be hyperkähler.

[1] We take the gauge group to be noncompact, i.e. $\mathbb{R}^n$ rather than $\mathrm{U}(1)^n$. Since we take all hypermultiplets to have unit charge, the coefficients k_{ij} must be rational in the compact case; then there is a basis where all k_{ij} and all hypermultiplet charges are integers. Note that Green's functions which are well-defined in the non-compact case are identical in the compact case and are continuous functions of k.

Consider first $\mathcal{N} = 3$ SQED-1 with a CS coupling k and infinite bare gauge coupling. In this case there is no moduli space: the Coulomb branch is lifted because the CS term gives the vector multiplet a topological mass, while the Higgs branch is lifted because integrating out the vector multiplet produces a potential for the hypermultiplet. When $k \to \infty$ the vector multiplet decouples, and the theory becomes a theory of a free massless hypermultiplet, with $\mathcal{N} = 4$ supersymmetry and $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_N$ R-symmetry. Since the coefficient of the CS term is not renormalized [14, 15], the theory with $k \neq \infty$ is an exactly marginal deformation of the free hypermultiplet, in analogy to the BF theory considered earlier. One can perform an ordinary Feynman diagram expansion in $1/k$, which by power counting and supersymmetry is completely finite. Since there are no dimensionful parameters, there is no wave-function renormalization of the hypermultiplet. This also follows from the fact that chiral gauge-invariant operators like $Q\tilde{Q}$ belong to short representations of $\mathcal{N} = 3$ superconformal algebra, and their dimension is determined entirely by their $\mathrm{SU}(2)_D$ spin via $\Delta = j_D$. The dimensions of nonchiral operators will generally depend on k.

In the opposite limit $k \to 0$ we obtain $\mathcal{N} = 4$ SQED-1 with infinite gauge coupling, i.e. CFT-1, which is mirror to the free hypermultiplet found as $k \to \infty$. As in the BF case, we are led to the conclusion that $\mathcal{N} = 3$ SQED-1 with CS coupling k is dual to $\mathcal{N} = 3$ SQED-1 with CS coupling $-1/k$. (The inversion of the CS coupling was previously argued, using branes in Type IIB string theory, in [5].) The generating functional of CFT-1 with CS coupling k is

$$Z_k[\mathcal{U}] = \int \mathcal{D}\mathcal{V}\ \mathrm{Sdet}\left(\mathcal{K}[\mathcal{V}]\right) \exp\left(ikS_{CS}(\mathcal{V},\mathcal{V}) + iS_{GF}(\mathcal{V}) + iS_{BF}(\mathcal{U},\mathcal{V})\right). \qquad (5.2)$$

Upon using eq. (3.4) and performing the Gaussian integral over $\mathcal{V}$ this becomes

$$Z_{-1/k}\left[-\frac{1}{k}\mathcal{U}\right] \exp\left(-\frac{i}{k}S_{CS}(\mathcal{U},\mathcal{U})\right). \qquad (5.3)$$

This shows that the connected correlators of $\mathcal{V}$ at CS coupling k and CS coupling $-1/k$ are related by a simple rescaling (and a shift by a contact term for the two-point function).

Our discussion can be easily generalized to theories with more multiplets. Consider N_f copies of CFT-1 and couple the vector multiplets together as in (5.1). The theories at the origin of moduli space make up a manifold of $\mathcal{N} = 3$ SCFTs parameterized by the matrix k. Mirror symmetry acts on this manifold by $k \to -k^{-1}$. For generic k there is no moduli space; the only solution of the classical D-flatness equations

$$\frac{1}{2\pi}\sum_j k_{ij}\Phi_j^p = Q_i^\dagger \sigma^p Q_i \equiv H_i^p\,, \qquad |\Phi_i|^2 |H_i|^2 = 0\,, \qquad (5.4)$$

(no sum on i) is the trivial one, $\Phi = Q = \tilde{Q} = 0$. In general, we may search for solutions as follows. The second set of equations in (5.4) requires that we divide the indices i into two sets, which without loss of generality (through relabeling) we may take to be $I = 1, 2, \ldots, n$ and $r = n+1, \ldots N_f$, and set $H^p_I = 0, \Phi^p_r = 0$. The equations

$$k_{IJ}\Phi^p_J = 0$$

have nontrivial solutions if k_{IJ} has zero determinant. If this is the case, then $\Phi^p_J = \sum_v c^p_v e^v_J$, where the e^v_J are the zero modes of the minor k_{IJ}, and c^p_v are three sets of coefficients, $p = 1, 2, 3$. If we expand the photons A^μ_J as $A^\mu_J = \sum_v b^\mu_v(x) e^v_J + \cdots$, then the fields $b^\mu_v(x)$ do not couple to themselves via CS terms, so their dual scalars τ_v may be defined in the usual manner. The fields $b^\mu(x)$ do couple to other photons, via BF-type terms. As a result of this the scalars τ_v transform additively under gauge transformations of other photons. Meanwhile, the other equations

$$H^p_r = \frac{1}{2\pi} k_{rJ}\Phi^p_J$$

fix the expectation values of Q_r up to a gauge transformation. As in the BF case, these equations can be interpreted as moment map equations for a hyperkähler quotient. When Maxwell terms are present, the corresponding hyperkähler metric is again of the LR/LWY type [17, 12]. An interesting issue is whether there are quantum corrections to this metric. The hyperkähler property of the metric and the presence of triholomorphic U(1) isometries (coming from the shift symmetries of the dual photons) ensure that the quantum metric remains of the LR/LWY type. However, these considerations still allow the parameters of the metric to depend on the elements of the matrix k in an arbitrarily complicated manner. We have not resolved this issue completely.

These results contain, as special cases, our results on $\mathcal{N} = 4$ theories with BF couplings and the mirror relations considered in [5]. Furthermore, the general $\mathcal{N} = 3$ CS theory can be reduced to this example by linear field redefinitions and possible addition of decoupled ($k = \infty$) vector and/or hypermultiplets.

It is interesting to note that duality of certain Chern-Simons theories with respect to the inversion of the CS coupling has been conjectured to underlie the structure of the phase diagram of quantum Hall liquids; see for example [18, 19, 20].

6. Piecewise mirror transformations

Up to this point we have limited ourselves to discussing mirror transformations applied to a theory as a whole, converting all ordinary multiplets to twisted multiplets and vice versa. However, nothing prevents us from applying the mirror transform, as given in eq. (3.4), to one hypermultiplet or twisted hypermultiplet at a time. We will call this operation a “piecewise mirror transform.” In general a theory with p hypermultiplets and q twisted hypermultiplets will have 2^{p+q} piecewise-mirror descriptions.

To illustrate this we consider the simplest non-trivial example. Take U(1) with two hypermultiplets, one of charge 1 and one of charge q. The usual mirror transform converts its infrared conformal field theory to that of twisted U(1) with two twisted hypermultiplets of charge 1 and $-1/q$. (Note the sign of a hypermultiplet charge can be changed by a field redefinition.) If instead we apply a piecewise mirror transform to the hypermultiplet of charge 1, using eq. (3.4), we will find a theory with the following content: a vector multiplet coupled to a twisted vector multiplet with a BF coupling $k = 1$, a hypermultiplet of charge q coupled to the vector multiplet, and a twisted hypermultiplet of charge 1 coupled to the twisted vector multiplet. Rescaling the vector multiplet, we may set the hypermultiplet charge to 1 and the BF coupling to $k = 1/q$. Thus, U(1) with hypermultiplets of charge $1, q$ is piecewise-mirror to the BF theory in eq. (4.1) with coupling $k = 1/q$.

If instead we apply the piecewise mirror transform to the hypermultiplet of charge q, we will similarly find a BF theory with coupling $k = -q$. This is the mirror of the previous BF theory. The following four theories are thus piecewise-mirror

$$\begin{array}{ccc} & \mathrm{BF}(\hat{\mathcal{Q}}, \hat{\mathcal{V}}, \mathcal{V}, \mathcal{Q})[k=-q] & \\ \text{CFT-2}(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{V})[q_1 = 1, q_2 = q] & & \text{CFT-2}(\hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_2, \hat{\mathcal{V}})[q_1 = 1, q_2 = -q^{-1}] \\ & \mathrm{BF}(\mathcal{Q}, \mathcal{V}, \hat{\mathcal{V}}, \hat{\mathcal{Q}})[k = q^{-1}] & \end{array}$$

Note that the self-duality (up to a sign) of U(1) with two hypermultiplets of equal charge is equivalent to the self-duality (up to a sign) of the BF theory with $k = 1$.

As a final comment, we note that the compactness of U(1) requires that the ratio of the charges of the hypermultiplets be rational. It follows from this, and from mirror symmetry, that both q and k must be rational. This is consistent with the condition on k that we mentioned earlier.

It is easy to apply the piecewise mirror transform to other models, including the Chern-Simons theories of the previous section and the non-conformal field theories of the next.

7. Mirror symmetry away from the infrared limit

In this section we give a field-theoretic interpretation of the so-called "magnetic coupling"[2] [1] and explain how the mirror transform can be extended away from the infrared limit. The "magnetic coupling" affects the metric on the Higgs branch, as will be reviewed below, and is mirror to the gauge coupling. However, its field theory origin has not previously been determined. As we will now show, it is a

[2]The term "magnetic coupling" is an unfortunate misnomer, as the relation between this interaction and the electric gauge interaction is not electric-magnetic duality. Mirror symmetry exchanges particles and vortices, which couple (in the absence of a Chern-Simons coupling) to different photons.

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Fermi-type coupling — that which is induced between (twisted) hypermultiplets by the exchange of a massive auxiliary (twisted) vector multiplet to which they are minimally coupled. We will refer to the theory of a single hypermultiplet coupled to this massive auxiliary vector multiplet as super-Fermi theory (SFT).

An indirect way to check that the gauge and super-Fermi couplings are mirror, and that the super-Fermi coupling is indeed the constant term in the metric on the Higgs branch, is to consider SQED-2 (with fields $\mathcal{V}$, $\mathcal{Q}_1$, $\mathcal{Q}_2$) and its mirror of the same form (with twisted fields $\hat{\mathcal{V}}$, $\hat{\mathcal{Q}}_1$, $\hat{\mathcal{Q}}_2$). We will take the bare electric and "magnetic" couplings to be infinite.

The Coulomb branch is parameterized by the $\mathrm{SU}(2)_N$ triplet $\vec{\Phi}$ and the scalar τ which is the electromagnetic dual of the photon. The metric is specified in terms of a harmonic function $G(\vec{\Phi})$

$$ds^2 = G(\vec{\Phi})(d\vec{\Phi}^2) + G^{-1}(\vec{\Phi})(d\tau + \omega \cdot d\vec{\Phi})^2 \,, \tag{7.1}$$

where $\nabla \times \omega(\vec{\Phi}) = \nabla G(\vec{\Phi})$. In the presence of a mass term $\vec{m}$ (a triplet of $\mathrm{SU}(2)_N$) for $\mathcal{Q}_1$ and a mass term $-\vec{m}$ for $\mathcal{Q}_2$ the function G is given by

$$G = \frac{1}{|\vec{\Phi} - \vec{m}|} + \frac{1}{|\vec{\Phi} + \vec{m}|} \,. \tag{7.2}$$

We may obtain SQED-1 by integrating out $\mathcal{Q}_2$, i.e. by taking m large while keeping $\vec{\phi} = \vec{\Phi} - \vec{m}$ fixed. In this limit we get

$$G \approx \frac{1}{|\vec{\phi}|} + \frac{1}{2|\vec{m}|} \,. \tag{7.3}$$

The constant term in G is the gauge coupling induced at one-loop by integrating out the massive field $\mathcal{Q}_2$; the one-loop integral leads to a Maxwell term $\frac{1}{2m} S_V(\mathcal{V})$. The low-energy theory is SQED-1 with an effective coupling $g_{eff}^2 = 2m$.

In the mirror theory, the same branch appears as the Higgs branch, which is parameterized by three fields $\vec{N}$, triplets of $\mathrm{SU}(2)_N$ which are bilinear in the twisted hypermultiplets, along with a fourth scalar whose relation to the underlying fields is more complex. The metric on the Higgs branch similarly depends on a harmonic function $\hat{G}$ of $\vec{N}$ and a possible Fayet-Iliopoulos parameter $\vec{\xi}$ which is mirror to the mass term $\vec{m}$:

$$\hat{G} = \frac{1}{|\vec{N} - \vec{\xi}|} + \frac{1}{|\vec{N} + \vec{\xi}|} \,. \tag{7.4}$$

The mirror of taking $\vec{m} = \vec{\Phi} - \vec{\phi}$ is to take $\vec{\xi} = \vec{N} - \vec{n}$. For $|\vec{\xi}| \approx |\vec{N}| \gg |\vec{n}|$, the field $\hat{\mathcal{Q}}_2$ condenses and gives mass to $\hat{\mathcal{V}}$, leaving the field $\hat{\mathcal{Q}}_1$ behind. In the limit where ξ is large $\hat{G}$ becomes

$$\hat{G} = \frac{1}{|\vec{n}|} + \frac{1}{2|\vec{\xi}|} \,. \tag{7.5}$$

What is the interpretation of the constant term in $\hat{G}$? It must be the coupling of the leading dimension-four operator induced in this broken gauge theory — which is obviously the super-Fermi interaction for $\mathcal{Q}_1$ induced by the massive photon.[3]

A more direct argument involves the computation of the metric in the presence of the super-Fermi interaction. To give a precise definition of this interaction, let us multiply both sides of eq. (3.4) by $\exp(iS_V(\hat{\mathcal{V}})/g^2)$ and integrate over $\hat{\mathcal{V}}$. The left hand side becomes the partition function of twisted $\mathcal{N}=4$ SQED-1 with bare gauge coupling g, while the right-hand side corresponds to $\mathcal{N}=4$ SQED-1 with infinite bare gauge coupling coupled via a BF term to a twisted vector multiplet $\hat{\mathcal{V}}$. The action of the latter theory is

$$S = S_H(\mathcal{Q},\mathcal{V}) + S_{BF}(\hat{\mathcal{V}},\mathcal{V}) + \frac{1}{g^2}S_V(\hat{\mathcal{V}})\,. \tag{7.6}$$

This is what we call the super-Fermi theory (SFT). Since the action for $\hat{\mathcal{V}}$ is quadratic it can be integrated over, leaving the action $S = S_H(\mathcal{Q},\mathcal{V}) + g^2 S_{Vaux}(\mathcal{V})$ with

$$S_{Vaux}(\mathcal{V}) = -\frac{1}{4\pi^2}\int d^3x \int d^4\theta \,\left\{\frac{1}{4}\Sigma\frac{1}{\Box}\Sigma - \Phi^\dagger\frac{1}{\Box}\Phi\right\}\,, \tag{7.7}$$

which in the Landau gauge becomes an explicit mass term for $\mathcal{V}$,

$$S_{Vaux}(\mathcal{V}) = -\frac{1}{4\pi^2}\int d^3x \int d^4\theta \,\left\{V^2 - \Phi^\dagger\frac{1}{\Box}\Phi\right\}. \tag{7.8}$$

Thus $\mathcal{V}$ acts as an auxiliary field at the classical level. After integrating it out, we find by direct if tedious computation that the action for $\mathcal{Q}$ is that of a sigma-model with the Taub-NUT target space. Moreover, the asymptotic radius of the circle parameterized by τ agrees with that computed from the mirror SQED-1 theory. (Another way of doing the same computation, using hyperkähler quotients, was explained in section 4.) The hyperkähler property of the metric ensures that there are no quantum corrections to this result.

So far we showed the moduli space metrics of SQED-1 with finite gauge coupling and twisted SFT agree, i.e. that the two theories are equivalent in the extreme infrared everywhere on the moduli space. We now claim that this equivalence is exact, so that $\mathcal{N}=4$ SQED-1, in its renormalization group flow from weak to strong coupling, is mirror to twisted SFT at all energy scales.

This seems to be a very strong claim, as most known field theoretic dualities have been established only in the infrared or for conformal field theories. However,

[3]Note that string theory considerations also support this claim. A D3 brane of finite length L which ends on two parallel NS5 (D5) branes contains as its lightest multiplets a massless $\mathcal{N}=4$ U(1) vector multiplet (hypermultiplet) and a massive hypermultiplet (vector multiplet) of mass of order $\sim 1/L$ [7]. The gauge coupling of the vector multiplet in the NS5 case is also of order $1/L$, and so the gauge coupling in one theory is related to the mass of a vector multiplet in its mirror.

if one has two well-defined exact descriptions of an ultraviolet fixed point, then all perturbations of this fixed point and the resulting renormalization group flows can be described using the two sets of variables. This is the case here. The ultraviolet fixed point of which SQED-1 is a perturbation is a free theory of a hypermultiplet and a vector multiplet which are not coupled to one another. This CFT has a mirror description as a copy of twisted CFT-1 along with a vector multiplet to which it is not coupled. Consider the relevant perturbation given on one side by coupling the vector multiplet to the flavor current of the hypermultiplet, and on the other side by coupling the vector multiplet to the global current ${}^*\hat{F}$ of the twisted CFT-1 via a BF term. This makes the first theory into SQED-1 with a weak gauge coupling and the second into a theory of a twisted hypermultiplet coupled to an auxiliary vector multiplet, which induces a large super-Fermi coupling. The gauge coupling in SQED-1 grows, and in the infrared the theory becomes CFT-1. The super-Fermi coupling in the mirror theory shrinks, and in the infrared the twisted hypermultiplet becomes free. To restate the claim, mirror symmetry implies

$$\begin{array}{llcl}
UV: & \text{free } \mathcal{Q} \ + \ \text{free } \mathcal{V} & \Longleftrightarrow & \text{twisted CFT-1}(\hat{\mathcal{Q}},\hat{\mathcal{V}}) + \text{free } \mathcal{V} \\
\Downarrow & \Downarrow & & \Downarrow \\
\text{flows to} & \text{SQED-1}(\mathcal{Q},\mathcal{V}) \ [\text{coupling } g^2] & \Longleftrightarrow & \text{twisted SFT}(\hat{\mathcal{Q}},\hat{\mathcal{V}},\mathcal{V}) \ [\text{coupling } 1/g^2] \\
\Downarrow & \Downarrow & & \Downarrow \\
IR: & \text{CFT-1}(\mathcal{Q},\mathcal{V}) & \Longleftrightarrow & \text{free twisted } \hat{\mathcal{Q}}
\end{array} \tag{7.9}$$

A corollary of this equivalence is that all correlation functions of Σ and Φ in SQED-1 must precisely agree with those of the U(1) current multiplet in the twisted SFT. This can be seen explicitly from our master equation eq. (3.4). To this end multiply both sides of eq. (3.4) by

$$\exp\left\{\frac{i}{g^2}S_V(\hat{\mathcal{V}}) + iS_{BF}(\hat{\mathcal{V}},\mathcal{V}') + iS_{GF}(\hat{\mathcal{V}})\right\}$$

and integrate over $\hat{\mathcal{V}}$. After performing a Gaussian integral over $\hat{\mathcal{V}}$ on the right-hand side and shifting the integration variables, one gets

$$\begin{aligned}
\int \mathcal{D}\hat{\mathcal{V}} \quad \exp\left\{\frac{i}{g^2}S_V(\hat{\mathcal{V}}) + iS_{GF}(\hat{\mathcal{V}}) + iS_{BF}(\hat{\mathcal{V}},\mathcal{V}')\right\} \text{Sdet}\left(\mathcal{K}[\hat{\mathcal{V}}]\right) = \\
= \int \mathcal{D}\mathcal{V} \ \exp\left\{ig^2 S_{V aux}(\mathcal{V}) + iS_{GF}(\mathcal{V})\right\} \text{Sdet}\left(\mathcal{K}[\mathcal{V}-\mathcal{V}']\right).
\end{aligned} \tag{7.10}$$

Here the left-hand side is the generating functional for the correlation functions of $\hat{\Sigma}$ and $\hat{\Phi}$ in (twisted) SQED-1, while the right-hand side is the generating functional for the correlators of the hypermultiplet's U(1) flavor current in SFT.

An interesting implication of this result is that the perturbative expansion of equation (7.10) in g is the superrenormalizable SQED expansion around a free theory, while the perturbation series in $1/g$ is the usual nonrenormalizable SFT expansion around a free theory. The former expansion is finite, while the latter requires renormalization and fails in the ultraviolet. We see that despite the failure of the usual perturbative expansion in SFT, the theory still has a perfectly well defined UV fixed point, as in the five and six dimensional field theories considered first in [21, 22].

It is also instructive to consider the current-current correlation function in SQED. For example, consider SQED-1 with a non-zero Fayet-Iliopoulos term $\vec{\xi}$, which gives the hypermultiplet an expectation value, $\langle Q^\dagger \vec{\sigma} Q\rangle = \vec{\xi}/(2\pi)$. If $|\xi| \gg g^2$, the photon is massive ($m^2_{\mathcal{V}} \approx g^2\xi/\pi$) and stable, and shows up as a single particle state in the two-point function of ${}^*F$. In addition there are much heavier semiclassical vortex states, with $m^2_{\hat{\mathcal{Q}}} = \xi^2$, which can be pair-produced by the current. Note that the vortex mass is protected by a BPS bound while that of the photon is not. As we reduce $|\xi|/g^2$, the photon and vortex masses approach each other. It is possible, for sufficiently small $|\xi|$, that the photon becomes unstable and decays into vortices, leaving no stable one-particle states in this channel. Does this occur?

For small $|\xi|$ the original variables are strongly coupled, so we must use the mirror variables, which describe massive vortices of mass $|\xi|$ weakly interacting via a short-distance potential. The potential energy of a configuration of vortices is zero, but for a configuration of both vortices and antivortices it is negative. It is known that two non-relativistic particles with an attractive delta-function potential in two spatial dimensions have a single bound state with exponentially small binding energy [23]. We therefore expect a single irreducible supermultiplet of stable vortex-antivortex bound states. This supermultiplet is an ordinary $\mathcal{N}=4$ vector multiplet. It therefore appears that there is a stable massive vector multiplet in the theory for any value of $|\xi|/g^2$, only merging into the continuum of vortex-antivortex states at $\xi = 0$. We believe this is a new result that could not have been derived without the identification of the magnetic coupling.

Let us find the binding energy of this bound state. The coefficient of the delta function in the low-energy non-relativistic theory is logarithmically divergent. To obtain a sensible result we must match it to the coefficient of the Fermi interaction in the relativistic SFT theory. Supersymmetry ensures the relativistic Fermi interaction receives only finite corrections, which are small if $|\xi| \ll g^2$. Matching requires a cutoff, which should be at the scale of the breakdown of the non-relativistic theory, that is, of order $|\xi|$. Putting this together with the known result [23], we find the binding energy is of order $-|\xi|e^{-g^2/2\pi|\xi|}$.

The results of this section can be easily extended to theories with more flavors. We mention one amusing example with a self-mirror renormalization group trajectory. Consider CFT-2, the infrared limit of SQED-2, which is self-mirror. The theory has two global symmetry currents, a flavor current and a topological current exchanged under mirror symmetry. Using a vector multiplet $\mathcal{V}$ and a twisted vector multiplet $\hat{\mathcal{V}}$, we may gauge both currents with equal couplings. The resulting theory

$$\int \mathcal{D}\mathcal{V}_0 \, \mathcal{D}\hat{\mathcal{V}} \, \mathcal{D}\mathcal{V} \, e^{\frac{i}{g^2}[S_V(\mathcal{V})+S_V(\hat{\mathcal{V}})]} e^{iS_{BF}(\hat{\mathcal{V}},\mathcal{V}_0)} \, \mathrm{Sdet}\left(\mathcal{K}[\mathcal{V}_0+\mathcal{V}]\right) \, \mathrm{Sdet}\left(\mathcal{K}[\mathcal{V}_0-\mathcal{V}]\right), \tag{7.11}$$

(here gauge fixing terms and couplings to background sources are omitted for brevity) flows from CFT-2 plus free vector and twisted vector multiplets in the ultraviolet to CFT-2 in the infrared. The flow can easily be seen, using eq. (3.4), to be self-mirror at all scales.

8. Vortex-creation operators

Up to this point our discussion has been mostly concerned with the action of mirror symmetry on conserved currents and their superpartners. But if we want to make precise the statement that mirror symmetry exchanges particles and vortices [10], we need to understand vortex-creation operators in SQED.

The gauge-invariant vortex-creation operators are associated with some of the most poorly understood aspects of mirror symmetry. Mirror symmetry unambiguously implies that such operators must be present in the CFTs that are found at the origin of moduli space. However, the only hint as to how to define them is found far along the moduli space of the Coulomb branch, where all of the charged matter is massive. There, the low-energy theory involves only the vector multiplet, and one may safely replace each photon with its dual scalar τ. Vortex creation operators are known to be proportional to $e^{i\tau}$. From mirror symmetry we know that some of the vortex-creation operators are chiral, in the $\mathcal{N}=2$ sense, and so, if the real scalar ϕ in the $\mathcal{N}=2$ vector multiplet has an expectation value, a natural form for a vortex-creation operator is $e^{(i\tau+\phi/g^2)}$, where g^2 is the low-energy effective gauge coupling. However, a number of puzzles surround this choice. How are these operators to be continued to the origin of moduli space, where there is a CFT involving massless charged matter which prevents naive definition of τ? What is to be done about the paradox that ϕ remains dimensionful in the CFT but no scale g^2 remains in the theory? Assuming these problems are resolved, how does the operator obtain its correct conformal dimension? How does it acquire its abelian global charges? For $\mathcal{N}=4$ SQED, where this operator should be part of an $\mathrm{SU}(2)_N$ multiplet, what are the other operators in the multiplet and how is the nonabelian global symmetry realized? For those cases where there are hidden flavor symmetries [1, 10] which must act on these operators, how do those symmetries appear?

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We will now attempt to provide answers to some of these questions. To set the stage, let us recall how to construct operators with nonzero vortex charge in an arbitrary abelian gauge theory in three dimensions (formally). Consider a U(1) gauge field coupled to massless matter. Formally integrating over matter fields, we get a nonlocal effective action for the gauge field A^μ. Gauge invariance tells us that it can be regarded as a functional $S_{eff}(F)$ of the field strength $F = dA$. Let us change variables in the path integral from A^μ to $F^{\mu\nu}$; since $F^{\mu\nu}$ satisfies the constraint $dF = 0$, we must implement it using a Lagrange multiplier τ:

$$\int \mathcal{D}A^\mu \; \delta(\partial_\mu A^\mu) e^{iS_{eff}\{F(A)\}} \propto \int \mathcal{D}F^{\mu\nu} \; \mathcal{D}\tau \; \exp\left(iS_{eff}(F) + \frac{i}{2\pi} \int d^3x \; \tau dF \right). \tag{8.1}$$

Note ${}^*F$ has dimension 2 (as demanded of a conserved current by the conformal algebra) so τ is dimensionless and, in analogy to a free boson in two dimensions, can be exponentiated. It follows from (8.1) that τ is canonically conjugate to $\epsilon^{ij} F_{ij}$, so that the symmetry transformation generated by the current ${}^*F$ acts additively on τ and multiplicatively on $e^{in\tau}$. Our normalization is such that $e^{i\tau}$ carries a unit of this charge, the integrated magnetic flux.

Since a vortex worldline carries magnetic flux, any operator which creates a vortex must be proportional to $e^{i\tau}$ [10]. To see this, consider the correlation function of two such operators

$$\int \mathcal{D}F^{\mu\nu} \mathcal{D}\tau \exp\left\{ iS_{eff}(F) + \frac{i}{2\pi} \int d^3x \tau dF \right\} e^{i\tau(x)} e^{-i\tau(y)} . \tag{8.2}$$

The integration over τ gives a factor of $\delta[dF - 2\pi\delta(x) + 2\pi\delta(y)]$. Thus, the Bianchi identity is violated by two pointlike sources of magnetic flux — pointlike nondynamical Dirac monopoles, which are instantons in three dimensions. On the Higgs branch, where flux is confined into particle-like vortex solitons, these pointlike instantons will indeed be sources for these solitons.

A SUSY-covariant extension of this procedure can be constructed following [17], where it was shown how to dualize an $\mathcal{N} = 2$ vector multiplet to a chiral multiplet on the Coulomb branch of the moduli space. The superspace effective action for the U(1) vector multiplet V is regarded as a functional of $\Sigma = iD\bar{D}V$. Σ satisfies supersymmetric Bianchi identities $D^2\Sigma = 0 = \bar{D}^2\Sigma$. If we impose this constraint explicitly by introducing a Lagrange multiplier chiral superfield T which couples to Σ via $\int d^3x d^4\theta \; \Sigma(T + T^\dagger)$, then we may replace integration over V by integration over an unconstrained real superfield Σ. The partition function for an $\mathcal{N} = 2$ theory takes the form

$$\int \mathcal{D}\Sigma \mathcal{D}T \exp\left(iS_{eff}(\Sigma) + \frac{i}{4\pi} \int d^3x d^4\theta \; \Sigma(T + T^\dagger) \right). \tag{8.3}$$

The normalization in (8.3) is such that the imaginary part of the lowest component of T is τ, so e^T has vortex charge $+1$.

Consider now $\mathcal{N}=2$ or $\mathcal{N}=4$ SQED-N_f. The mirror of $\mathcal{N}=2$ SQED-N_f differs from that of $\mathcal{N}=4$ SQED-N_f, which was described in sec. 2, only by the presence of an extra neutral chiral superfield which couples to $\sum_p \hat{Q}_p\hat{\tilde{Q}}_p$ [9, 10]. In both cases the mirror $\mathrm{U}(1)^{N_f-1}$ gauge theory has $\mathcal{N}=2$ chiral primary operators $V_+ = \hat{Q}_1 \dots \hat{Q}_{N_f}$ and $V_- = \hat{\tilde{Q}}_1 \dots \hat{\tilde{Q}}_{N_f}$. Their vortex charges are $+1$ and -1, respectively. They live in a short representation of the $\mathcal{N}=2$ or $\mathcal{N}=4$ superconformal algebra, and therefore their dimensions are related to their R-charges. In $\mathcal{N}=4$ SQED-N_f the dimensions are fixed to be the canonical dimension, $N_f/2$. In $\mathcal{N}=2$ SQED-N_f the dimensions are not known, since the theory has a one-parameter family of R-currents from which it is not clear how to select the relevant one, but in the large N_f limit the R-charges and dimensions can be determined using mirror symmetry, as we will now explain.

As is well-known, non-supersymmetric QED is completely solvable in this limit (see [24]). The effective action for the photon given by integrating out N_f massless electrons is simply $\mathcal{L}_{eff} \propto N_f F^{\mu\nu}[-\Box]^{-1/2}F_{\mu\nu}$ plus higher orders in the field strength. As always, in the large N_f limit all scattering is suppressed and the theory becomes Gaussian; since the photon propagator is nonstandard, it is known as a "generalized free field."[4] Similarly, for $\mathcal{N}=4$ SQED in the large N_f limit one gets

$$\begin{aligned} Z_{N_f}[\hat{\mathcal{V}}] &= \int \mathcal{D}\mathcal{V}\; e^{iS_{BF}(\hat{\mathcal{V}},\mathcal{V})+iS_{GF}(\mathcal{V})}\;\left[\mathrm{Sdet}\,(\mathcal{K}[\mathcal{V}])\right]^{N_f} \\ &\approx \int \mathcal{D}\mathcal{V} e^{iS_{BF}(\hat{\mathcal{V}},\mathcal{V})+iS_{GF}(\mathcal{V})} \exp\left\{\frac{iN_f}{16}\int d^3x \int d^4\theta \left(\Sigma[-\Box]^{-\frac{1}{2}}\Sigma - 4\Phi^\dagger[-\Box]^{-\frac{1}{2}}\Phi\right)\right\}, \end{aligned} \tag{8.4}$$

with an analogous expression for $\mathcal{N}=2$ SQED. Thus the vector multiplet is described by a supersymmetric generalized free field. The dimensions of matter fields $Q, \tilde{Q}$ in SQED-N_f are canonical up to corrections of order $1/N_f$ [24], so the mesons $\tilde{Q}_pQ_p$ and their mirrors S_p [9, 10] have dimension 1. It follows that the dimensions of $\hat{Q}_p$ and $\hat{\tilde{Q}}_p$, which couple to S_p in the superpotential $W = S_p\hat{Q}_p\hat{\tilde{Q}}_p$, are canonical, so V_+ and V_- both have dimension $N_f/2$.

Consider now operators e^T and e^{-T} in $\mathcal{N}=2$ or $\mathcal{N}=4$ SQED-N_f, where T is the dual photon superfield defined above. These operators are (naively) chiral and have vortex charge $+1$ and -1, and the operators V_+ and V_- should therefore be proportional to them. The dimensions of $e^{\pm T}$ match those of $V_\pm$ in the large N_f limit, as we now show by computing the two-point function of the lowest component

[4]In position space the photon propagator is proportional to $1/x^2$. It was pointed out long ago that this is the same as the four-dimensional photon propagator projected down onto a three-dimensional hyperplane. We may observe that it is also the projection onto the boundary of four-dimensional Anti-de Sitter space of a photon propagating on that space. To be more precise, for a background gauge field coupled to the three-dimensional current ${}^*F$, the induced propagator in three-dimensions will also be $[-\Box]^{-1/2} \sim 1/x^2$, as though it were a free massless vector field on AdS_4. This is not to suggest large N_f (S)QED has a (super)gravity dual; the form of the propagator is fixed by conformal invariance alone.

of $e^{\pm T}$. Using (8.4) and performing the Gaussian integral over Σ we find:

$$\langle e^{T(x)} e^{T^\dagger(y)} \rangle \sim \int \mathcal{D}T \exp\left(T(x) + T^\dagger(y) - \frac{i}{2\pi^2 N_f} \int d^3z d^3z' T(z) [-\Box]^{3/2} T^\dagger(z') \right). \tag{8.5}$$

T is dimensionless and its propagator $[-\Box]^{-3/2}$ is logarithmic in position space, so the operator e^T has a well-defined dimension. Performing the Gaussian integral over T in (8.5) we find that e^T has dimension $N_f/2$.

In summary, we have clarified several issues. The field τ can still be defined at the origin of the moduli space without difficulty, as long as one first integrates out the massless charged matter and re-expresses the resulting non-local action S_{eff} using $F_{\mu\nu}$. The complex scalar which is exponentiated is

$$T|_{\theta=0} = i\tau - \frac{1}{8\pi} \frac{\delta S_{eff}}{\delta \Sigma}\Big|_{\theta=0},$$

a non-local expression which nonetheless agrees with expectations far along the Coulomb branch. With proper normalization, the dimensions of the vortex operators $e^{\pm T}$ have been shown to match those of $V_\pm$ in the large N_f limit, where S_{eff} can be computed.

However, this is not the whole story. Apart from their vortex charge, the operators $V_\pm$ carry non-zero and equal abelian R-charges. This is connected with the fact that in $\mathcal{N}=4$ SQED-N_f there is an operator relation of the form $V_+ V_- \sim \Phi^{N_f}$. It is impossible for e^T and e^{-T} to satisfy these constraints. Even more confusing is the fact that in $\mathcal{N}=4$ SQED-N_f the operators V_+ and $V_-^\dagger$ actually belong to a spin-$N_f/2$ multiplet of $SU(2)_N$.

To resolve these issues, care should be taken in the definition of the operators $e^{\pm T}$. As in two dimensions, the presence of a logarithmic propagator implies the need for an infrared regulator, which should be supersymmetry-preserving. This regulator may carry global symmetry charges, which might resolve some of the remaining puzzles. It is also possible that one must account for fermionic zero-modes of the pointlike Dirac monopoles that $e^{\pm T}$ are intended to represent.

As a last comment, we note that one of the most important unsolved problems in mirror symmetry is the mapping of the full nonabelian flavor symmetries. $\mathcal{N}=4$ SQED-N_f has an $SU(N_f)$ flavor symmetry, but in the mirror description only the diagonal generators are visible classically, with the rest emerging through quantum effects [1]. This can be seen from the fact that operators in nontrivial representations of $SU(N_f)$ appear in the mirror theory as a combination of operators built from fundamental fields with other operators built from vortex-creation operators [10]. A proper definition of the vortex-creation operators is a prerequisite for an understanding of the hidden symmetries.

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9. Outlook

We have found an elegant formula, eq. (3.4), which summarizes many known results of mirror symmetry. The formula states that the superdeterminant of the $\mathcal{N}=4$ supersymmetric Laplacian on three-dimensional Minkowski space is its own generalized Fourier transform. We have used it to find new superconformal field theories with exactly marginal couplings, on which mirror symmetry acts as strong-weak coupling duality. We have established mirror relations between non-conformal theories which are valid at all energy scales. Finally, we have made some progress toward understanding how to define the vortex-creation operators which appear in these theories. However, many questions remain. We do not have the precise change of variables underlying mirror symmetry, which requires a clearer understanding of vortex operators. We have no proof of our formula from first principles, and see no hint of a reformulation of the theory in which it would be manifest. Lastly, we have no idea how to generalize it to non-abelian gauge theories. We hope that future research will overcome these obstacles to a more profound understanding of duality.

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